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Generalizations of vector quasivariational inclusion problems with set-valued maps

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Abstract Existence theorems are given for the problem of finding a point (z_0, x_0) of a set $E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ and, for all $\eta \in A(z_0, x_0)$, $(F(z_0, x_0, x_0, \eta), C(z_0, x_0, x_0, \eta)) \in \alpha$ where α is a relation on 2^Y (i.e., a subset of $2^Y \times 2^Y$), $A : E \times K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $C : E \times K \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \times K \longrightarrow 2^Y$ are some set-valued maps, and Y is a topological vector space. Detailed discussions are devoted to special cases of α and C which correspond to several generalized vector quasi-equilibrium problems with set-valued maps. In such special cases, existence theorems are obtained with or without pseudomonotonicity assumptions.

Keywords Quasivariational inclusion problem · Set-valued map · Existence theorem · Pseudomonotonicity · Generalized concavity

1 Introduction

It is known [4] that equilibrium problem provides a unified approach to various problems in optimization theory, saddle point theory, game theory, fixed point theory, variational inequalities... In recent years, much attention is paid to generalized quasi-equilibrium problems. The generalized quasi-equilibrium problem in its simplest form is the problem of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \widehat{B}(x_0) \times \widehat{A}(x_0)$ and

$$\varphi(z_0, x_0, \eta) \geq 0, \quad \forall \eta \in \widehat{A}(x_0),$$

where E (resp. K) is a subset of a topological vector space Z (resp. X), $\widehat{A} : K \longrightarrow 2^K$ and $\widehat{B} : K \longrightarrow 2^E$ are set-valued maps, and $\varphi : E \times K \times K \longrightarrow \mathbb{R}$ is a function. The generalized

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quasi-equilibrium problem and its extensions are investigated in [5,6,20,21,23,28,37,41–43] under various assumptions. An existence result for such a problem is obtained in [5,Theorem 3.1] under the assumption that K is a convex compact set, f is a continuous function and $\widehat{A}$ is a set-valued map which is continuous in the sense of [3]. It is shown [41,Theorem 2] that the continuity property of f can be weakened to the lower semicontinuity property if $\widehat{A}$ belongs to a strictly smaller subclass $\mathcal{A}$ of the class of continuous set-valued maps. Namely, $\mathcal{A}$ consists of set-valued maps $\widehat{A}$ which are upper semicontinuous [3] and which have lower open sections, i.e., $\widehat{A}^{-1}(x) := \{\xi \in K : x \in \widehat{A}(\xi)\}$ is open in K for each $x \in K$. However, it is known [28,p. 178] that, for K being a convex compact set, each map $\widehat{A} \in \mathcal{A}$ is a constant (set-valued) map, i.e., $\widehat{A}$ does not depend on $\xi \in K$. So, Theorem 2 of [41] cannot be applied to non-constant set-valued maps $\widehat{A}$.

In [17] four generalized versions of the above quasi-equilibrium problem are introduced. More precisely, assuming that $\widehat{\varphi} : E \times K \times K \longrightarrow 2^Y$ and $\widehat{C} : K \longrightarrow 2^Y$ are set-valued maps, the authors of [17] consider the following problems $(\widehat{P_i})$, $i = 1, 2, 3, 4$:

Problem $(\widehat{P_i})$: Find $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \widehat{B}(x_0) \times \widehat{A}(x_0)$, and

$$\alpha_i(\widehat{\varphi}(z_0, x_0, \eta), \widehat{C}(x_0)), \quad \forall \eta \in \widehat{A}(x_0),$$

where α_i is a relation on 2^Y (i.e. α_i is a subset of $2^Y \times 2^Y$) defined by

$$\begin{aligned}
\alpha_1 &= \{(M, N) \in 2^Y \times 2^Y : M \not\subset N\},\\
\alpha_2 &= \{(M, N) \in 2^Y \times 2^Y : M \subset N\},\\
\alpha_3 &= \{(M, N) \in 2^Y \times 2^Y : M \cap N \neq \emptyset \text{ (the empty set)}\},\\
\alpha_4 &= \{(M, N) \in 2^Y \times 2^Y : M \cap N = \emptyset\},
\end{aligned}$$

and the symbol $\alpha_i(M, N)$ is used to denote that $(M, N) \in \alpha_i$.

Under the assumption that $\widehat{A}$ is a set-valued map having open lower sections, and under other suitable assumptions, several results for the existence of solutions of each of Problems $(\widehat{P_i})$ are established in [17,Theorems 3.1–3.5]. Unfortunately, as we will see in Sect. 3, the assumptions used in these theorems are not sufficient for the correctness of these results.

The aim of this paper is to give existence results for the following general model:

Problem (P_α) : Find a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ and

$$\alpha(F(z_0, x_0, x_0, \eta), C(z_0, x_0, x_0, \eta)), \quad \forall \eta \in A(z_0, x_0), \tag{1.1}$$

where α is a relation on the power set 2^Y of a topological vector space Y (i.e. α is a subset of $2^Y \times 2^Y$), E (resp. K) is a nonempty convex subset of a locally convex Hausdorff topological vector space Z (resp. X), $A : E \times K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $C : E \times K \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \times K \longrightarrow 2^Y$ are set-valued maps with nonempty values.

Observe that Problem (P_α) provides a general model which includes as special cases many known generalized scalar/vector quasi-equilibrium problems. Let us illustrate this remark by some recent problems which can be regarded as Problem (P_α) with appropriate choice of α, A, B, C and F.

(i) Each of Problems $(\widehat{P_i})$, $i = 1, 2, 3, 4$, which is formulated above and is studied in [17], corresponds to Problem (P_α) with $A(z, \xi) \equiv \widehat{A}(\xi)$, $B(z, \xi) \equiv \widehat{B}(\xi)$, $C(z, \xi, x, \eta) \equiv \widehat{C}(x)$, $F(z, \xi, x, \eta) \equiv \widehat{\varphi}(z, x, \eta)$ and $\alpha = \alpha_i$, $i = 1, 2, 3, 4$. Problems $(\widehat{P_i})$, $i = 1, 2, 3, 4$, include as special cases several known set-valued vector equilibrium problems investigated in [1,8,10,13,22,25,30,34,35,38].

(ii) The papers [39,43] deal with the following Problems $(\widetilde{P}_1)$ and $(\widetilde{P}_2)$:

Problem $(\widetilde{P}_1)$ (resp. Problem $(\widetilde{P}_2)$): Find $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times \widehat{A}(x_0)$ and

$$\begin{aligned} &\widetilde{\varphi}(z_0, x_0, \eta) \subset \widetilde{C}(z_0, x_0, x_0), \quad \forall \eta \in \widehat{A}(x_0), \\ \text{(resp.}\quad &\widetilde{\varphi}(z_0, x_0, x_0) \subset \widetilde{C}(z_0, x_0, \eta)), \quad \forall \eta \in \widehat{A}(x_0)) \end{aligned}$$

where $\widehat{A} : K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $\widetilde{C} : E \times K \times K \longrightarrow 2^Y$ and $\widetilde{\varphi} : E \times K \times K \longrightarrow 2^Y$ are set-valued maps. Obviously, Problem $(\widetilde{P}_1)$ is exactly Problem (P_α) with $A(z, \xi) \equiv \widehat{A}(\xi)$, $C(z, \xi, x, \eta) \equiv \widetilde{C}(z, \xi, x)$, $F(z, \xi, x, \eta) \equiv \widetilde{\varphi}(z, x, \eta)$ and $\alpha = \alpha_2$. Problem $(\widetilde{P}_2)$ is Problem (P_α) with $A(z, \xi) \equiv \widehat{A}(\xi)$, $C(z, \xi, x, \eta) \equiv \widetilde{C}(z, x, \eta)$, $F(z, \xi, x, \eta) \equiv \widetilde{\varphi}(z, \xi, x)$ and $\alpha = \alpha_2$.

(iii) Some existing papers (see e.g. [8,9,15] and references therein) consider simplified versions of the problem of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ and, for each $\eta \in A(z_0, x_0)$, there exists $y \in F(z_0, x_0, x_0, \eta)$ with $y \notin \widehat{C}(z_0, x_0, x_0, \eta)$ where A, B and F are as in the formulation of Problem (P_α) and $\widehat{C} : E \times K \times K \times K \longrightarrow 2^Y$ is a set-valued map. Since the existence of a point $y \in F(z_0, x_0, x_0, \eta)$ with $y \notin \widehat{C}(z_0, x_0, x_0, \eta)$ means that $F(z_0, x_0, x_0, \eta) \cap [Y \backslash \widehat{C}(z_0, x_0, x_0, \eta)] \neq \emptyset$, we see that the above problem is a special case of Problem (P_α) with $C(z, \xi, x, \eta) := Y \backslash \widehat{C}(z, \xi, x, \eta)$ and $\alpha = \alpha_3$. This explains that our general model includes as special cases all the problems mentioned in [8,9,15].

(iv) In [14,24] some simultaneous vector quasi-equilibrium problems are examined. More precisely, assume that $g_1 : K_1 \times K_2 \times K_1 \longrightarrow 2^{Y_1}$, $g_2 : K_1 \times K_2 \times K_2 \longrightarrow 2^{Y_2}$, $A_i : K_1 \times K_2 \longrightarrow 2^{K_i}$, $i = 1, 2$, and $C_i : K_1 \times K_2 \longrightarrow 2^{Y_i}$, $i = 1, 2$, are set-valued maps where Y_i, $i = 1, 2$, are topological vector spaces and K_i, $i = 1, 2$, are nonempty convex sets of topological vector spaces X_i, $i = 1, 2$. One of the problems considered in [24] is the following Problem $(\widehat{P}_{4,4})$:

Problem $(\widehat{P}_{4,4})$: Find a point $(x_1^0, x_2^0) \in K_1 \times K_2$ such that $x_i^0 \in A_i(x_1^0, x_2^0)$, $i = 1, 2$, and

$$g_1(x_1^0, x_2^0, \eta_1) \cap \operatorname{int} C_1(x_1^0, x_2^0) = \emptyset, \quad \forall \eta_1 \in A_1(x_1^0, x_2^0), \tag{1.2}$$

$$g_2(x_1^0, x_2^0, \eta_2) \cap \operatorname{int} C_2(x_1^0, x_2^0) = \emptyset, \quad \forall \eta_2 \in A_2(x_1^0, x_2^0), \tag{1.3}$$

where we assume that $\emptyset \neq \operatorname{int} C_i(\xi_1, \xi_2) \neq Y_i$ for all $i = 1, 2$ and $(\xi_1, \xi_2) \in K_1 \times K_2$. (The symbol "int" denotes the interior.)

Some approaches can be used to show that Problem $(\widehat{P}_{4,4})$ is a special case of Problem (P_α). Let us mention one of them. For this purpose, let us introduce the Cartesian products $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ and $K = K_1 \times K_2$. For $\xi = (\xi_1, \xi_2) \in K$, $x = (x_1, x_2) \in K$ and $\eta = (\eta_1, \eta_2) \in K$, let us set

$$\begin{aligned} F(z, \xi, x, \eta) &\equiv g_1(\xi_1, x_2, \eta_1) \times g_2(x_1, \xi_2, \eta_2) \subset Y, \\ C(z, \xi, x, \eta) &\equiv [Y_1 \backslash \operatorname{int} C_1(\xi_1, \xi_2)] \times [Y_2 \backslash \operatorname{int} C_2(\xi_1, \xi_2)] \subset Y, \\ A(z, \xi) &\equiv A_1(\xi_1, \xi_2) \times A_2(\xi_1, \xi_2) \subset K. \end{aligned}$$

Then obviously, conditions (1.2) and (1.3) can be rewritten as the unique requirement that

$$F(z_0, x_0, x_0, \eta) \subset C(z_0, x_0, x_0, \eta), \quad \forall \eta \in A(z_0, x_0),$$

where $x_0 := (x_1^0, x_2^0)$ and $\eta = (\eta_1, \eta_2)$. Now it is clear that Problem $(\widehat{P}_{4,4})$ can be regarded as Problem (P_α) if we define the relation α on $2^Y = 2^{Y_1 \times Y_2}$ as the family of all pairs $(M, N) \in 2^Y \times 2^Y$ such that the conditions $(M_1, N_1) \in \alpha_2$ and $(M_2, N_2) \in \alpha_2$ are simultaneously satisfied, where M_1 and M_2 (resp. N_1 and N_2) are the "components" of M (resp.

N), i.e., $M = M_1 \times M_2 \subset Y = Y_1 \times Y_2$ (resp. $N = N_1 \times N_2 \subset Y = Y_1 \times Y_2$). Recall that $(M_i, N_i) \in \alpha_2$ means that $M_i \subset N_i$. It is worth noticing that in our case the set E and the set-valued map B play no role in Problem (P_α) since by the above definitions F, C and A do not depend on z.

The symmetric vector quasi-equilibrium problem investigated in [14] is to find a point $(x_1^0, x_2^0) \in K_1 \times K_2 \subset X_1 \times X_2$ such that $x_i^0 \in A_i(x_1^0, x_2^0)$, $i = 1, 2$, and

$$\varphi_1(x_1^0, x_2^0) - \varphi_1(\eta_1, x_2^0) \notin \operatorname{int} C_1, \quad \forall \eta_1 \in A_1(x_1^0, x_2^0),$$
$$\varphi_2(x_1^0, x_2^0) - \varphi_2(x_1^0, \eta_2) \notin \operatorname{int} C_2, \quad \forall \eta_2 \in A_2(x_1^0, x_2^0),$$

where X_i, Y_i, K_i and A_i, $i = 1, 2$, are as above, $\varphi_i : K_1 \times K_2 \longrightarrow Y_i$, $i = 1, 2$, are single-valued maps and $C_i \subset Y_i$, $i = 1, 2$, are (constant) closed convex cones with nonempty interior ($C_i \neq Y_i$, $i = 1, 2$). As in the case of Problem $(\widehat{P}_{4,4})$, this problem of [14] can be regarded as Problem (P_α) if for each $\xi = (\xi_1, \xi_2) \in K = K_1 \times K_2$, $x = (x_1, x_2) \in K = K_1 \times K_2$ and $\eta = (\eta_1, \eta_2) \in K = K_1 \times K_2$, we set

$$\begin{aligned} F(z, \xi, x, \eta) &\equiv [\varphi_1(x_1, \xi_2) - \varphi_1(\eta_1, \xi_2)] \times [\varphi_2(\xi_1, x_2) - \varphi_2(\xi_1, \eta_2)], \\ C(z, \xi, x, \eta) &\equiv [Y_1 \backslash \operatorname{int} C_1] \times [Y_2 \backslash \operatorname{int} C_2]. \end{aligned}$$

(The set-valued map A and the relation α are as above.)

The main result of this paper is Theorem 3.1 which gives sufficient conditions for the existence of solutions of Problem (P_α) with arbitrary relation α. We will study in detail the special case when $\alpha = \alpha_i$, $i = 1, 2, 3, 4$, and C does not depend on η and can be expressed as the sum of a convex set and a convex cone (see the maps (3.13) and (3.14) below). In particular, we will obtain correct results for the vector quasi-equilibrium problems in [17] under assumptions quite different from those of [17] and we will see that some known results of [5,6,13,14,20,21,23,39,43] are special cases of our main result.

We conclude this introduction by observing that the assumption of existence of open lower sections of some set-valued maps is not used in proving our results. We refer the reader to the recent papers [16,32,37] where such an assumption is needed for investigating problems which can be viewed as special cases of Problem (P_α) or which are similar to Problem (P_α).

2 Preliminaries

Let X be a topological space. Each subset of X is a topological space with a topology induced by the given topology of X. In this paper, neighbourhoods of $x \in X$ are understood as open neighbourhoods, and they are denoted by $U(x), U_1(x), U_2(x), \ldots$ The symbols cl A, int A and co A are used to denote the closure, interior and convex hull of A.

We use the symbol $f : X \longrightarrow 2^Y$ to denote that f is a set-valued map between two topological spaces X and Y. Let im f and gr f be the image and graph of f :

$$\operatorname{im} f = f(X) = \bigcup_{x \in X} f(x),$$

$$\operatorname{gr} f = \{(x, y) \in X \times Y : y \in f(x)\}.$$

If $g : X \longrightarrow 2^Y$ is another set-valued map, then we write $f \subset g$ if $f(x) \subset g(x)$ for all $x \in X$. If $\psi : X \longrightarrow 2^Z$ is a set-valued map between topological spaces X and Z, then the map $V = f \times \psi : X \longrightarrow 2^{Y \times Z}$ is defined by $V(x) = f(x) \times \psi(x)$ for all $x \in X$.

We will use the continuity properties of set-valued maps in the usual sense of [3]. Namely, f is upper semicontinuous (usc) if for any $x \in X$ and any open set $N \supset f(x)$ we have $N \supset f(x')$ for all x' from some neighbourhood $U(x)$ of x. Map f is lower semicontinuous (lsc) if for any $x \in X$ and any open set N with $f(x) \cap N \neq \emptyset$ we have $f(x') \cap N \neq \emptyset$ for all x' from some neighbourhood $U(x)$ of x. Map f is continuous if it is both usc and lsc. If gr f is a closed (resp. open) set of $X \times Y$, then we say that f has closed (resp. open) graph. A map having closed graph is also called a closed map. Map f is compact-valued (resp. closed-valued) if for all $x \in X$ $f(x)$ is a compact (resp. closed) subset of Y. Map f is compact if im f is contained in a compact set of Y. Observe that if $f : X \longrightarrow 2^Y$ and $\psi : X \longrightarrow 2^Z$ are compact then $f \times \psi$ is compact. Map f is acyclic if it is usc and if, for all $x \in X$, $f(x)$ is nonempty, compact and acyclic. Here a topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. It is known that contractible spaces are acyclic; and hence, convex sets and star-shaped sets are acyclic. Observe that the Cartesian product of two acyclic sets is acyclic (see the Künneth formulae in [33]).

We will need the following fixed point theorem due to Park [36,Theorem 7].

Theorem 2.1 *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X. If $f : K \longrightarrow 2^K$ is a compact acyclic map, then f has a fixed point, i.e., there exists $x_0 \in K$ such that $x_0 \in f(x_0)$.*

Assume that α is a relation on 2^Y, i.e., α is a subset of the Cartesian product $2^Y \times 2^Y$. For two sets $M \in 2^Y$ and $N \in 2^Y$, we write $\alpha(M, N)$ if and only if $(M, N) \in \alpha$. Denote by $\overline{\alpha}$ the relation on 2^Y defined by $\overline{\alpha} = 2^Y \times 2^Y \setminus \alpha$. Then the symbol $\overline{\alpha}(M, N)$ means that $(M, N) \notin \alpha$. We now give some propositions for later use.

Proposition 2.1 *Let X be a Hausdorff topological vector space and let Y be a topological vector space. Let α and β be arbitrary relations on 2^Y. Let $a \subset X$ be a nonempty compact convex subset, and $f : a \times a \longrightarrow 2^Y$, $c : a \times a \longrightarrow 2^Y$, $g : a \times a \longrightarrow 2^Y$ and $d : a \times a \longrightarrow 2^Y$ be set-valued maps with nonempty values such that*

(i) For all $(x, \eta) \in a \times a$,

$$\alpha(f(x, \eta), c(x, \eta)) \Longrightarrow \beta(g(x, \eta), d(x, \eta)).$$

(ii) For all $\eta \in a$, the set

$$s(\eta) = \{x \in a : \beta(g(x, \eta), d(x, \eta))\}$$

is closed in a.

(iii) For all $x \in a$, the set

$$t(x) = \{\eta \in a : \overline{\alpha}(f(x, \eta), c(x, \eta))\}$$

is convex.

(iv) For all $x \in a$, $\alpha(f(x, x), c(x, x))$.

Then the set

$$\{x \in a : \beta(g(x, \eta), d(x, \eta)), \quad \forall \eta \in a\}$$

is nonempty.

Proof This is an easy consequence of the KKM-Lemma (see [11]) applied to the map $s : a \longrightarrow 2^a$ defined in (ii). Indeed, first observe that s has nonempty closed values. In addition, it is a KKM-map in the sense that

$$\mathrm{co}\,\{\eta_j,\ j=1,2,\ldots,n\} \subset \bigcup_{j=1}^{n} s(\eta_j)$$

for all points $\eta_j \in a,\ j = 1, 2, ..., n$, where n is an arbitrary positive integer. Indeed, otherwise there exist $\eta_j \in a,\ j = 1, 2, ..., n$, and $x \in \mathrm{co}\,\{\eta_j,\ j = 1, 2, ..., n\}$ such that

$$x \notin \bigcup_{j=1}^{n} s(\eta_j),$$

i.e.,

$$\overline{\beta}(g(x,\eta_j), d(x,\eta_j)), \quad \forall j = 1, 2, \ldots, n.$$

By condition (i) $\eta_j \in t(x)$, for all $j = 1, 2, ..., n$. Because of the convexity of $t(x)$ (see (iii)) this yields $x \in t(x)$, i.e., $\overline{\alpha}(f(x,x), c(x,x))$, a contradiction to (iv). Applying the KKM-Lemma (see [11]) proves that there exists a point $x_0 \in a$ such that $x_0 \in s(\eta)$ for all $\eta \in a$. The conclusion of Proposition 2.1 is thus established. □

Remark 2.1 When a is not compact, Proposition 2.1 remains true under the following coercivity condition: there exist a nonempty compact set $a_1 \subset a$ and a compact convex set $b \subset a$ such that, for all $x \in a \backslash a_1$, there exists $\eta \in b$ with $\overline{\beta}(g(x,\eta), d(x,\eta))$.

Remark 2.2 When $\beta = \alpha_i$ (resp. $\overline{\alpha} = \alpha_i$), $i = 1, 2, 3, 4$, sufficient conditions for the validity of condition (ii) (resp. (iii)) of Proposition 2.1 are given in Corollary 3.2 below (resp. Proposition 2.3 below).

Proposition 2.2 *Let α and β be arbitrary relations on 2^Y. Let $a \subset X$ be a nonempty convex subset, and let $f, c, g, d : a \times a \longrightarrow 2^Y$ be set-valued maps with nonempty values. Assume that*

(i) For all $(x,\eta) \in a \times a$ with $x \neq \eta$, if $\overline{\alpha}(f(x,\eta), c(x,\eta))$ then $\overline{\alpha}(f(u,\eta), c(u,\eta))$ for some $u \in]x,\eta[$.
(ii) For all $(x,\eta) \in a \times a$ with $x \neq \eta$ and for all $u \in]x,\eta[$, if $\beta(g(x,u), d(x,u))$ then $\alpha(f(u,\eta), c(u,\eta))$.
(iii) For all $x \in a$, $\alpha(f(x,x), c(x,x))$.

Then

$$\{x \in a : \beta(g(x,\eta), d(x,\eta)), \quad \forall \eta \in a\} \subset \{x \in a : \alpha(f(x,\eta), c(x,\eta)), \quad \forall \eta \in a\}.$$

Proof Assume to the contrary that the conclusion of Proposition 2.2 is not true. Then there exists $x \in a$ such that

$$\forall \eta \in a, \quad \beta(g(x,\eta), d(x,\eta)) \tag{2.1}$$

and

$$\exists \xi \in a, \quad \overline{\alpha}(f(x,\xi), c(x,\xi)).$$

From this and from (iii) it follows that $x \neq \xi$. Hence, from (i), $\overline{\alpha}(f(u,\xi), c(u,\xi))$ for some $u \in]x,\xi[$. By (ii) $\overline{\beta}(g(x,u), d(x,u))$. This contradicts (2.1) since $u \in a$. □

Remark 2.3 Condition (ii) of Proposition 2.2 holds if condition (iii) of Proposition 2.2 is satisfied and if

(ii)′ For all $(x, \eta) \in a \times a$ with $x \neq \eta$ and for all $u \in]x, \eta[$, if $\beta(g(x, u), d(x, u))$ and $\overline{\alpha}(f(u, \eta), c(u, \eta))$ then $\overline{\alpha}(f(u, u), c(u, u))$.

Indeed, if condition (ii) of Proposition 2.2 does not hold, then by (ii)′ we have $\overline{\alpha}(f(u, u), c(u, u))$, a contradiction to condition (iii) of Proposition 2.2.

Remark 2.4 Condition (i) of Proposition 2.2 holds if for all $(x, \eta) \in a \times a$ with $x \neq \eta$ the set

$$\{u \in [x, \eta] : \overline{\alpha}(f(u, \eta), c(u, \eta))\} \tag{2.2}$$

is open in $[x, \eta]$.

Remark 2.5 From Corollary 3.2 to be given in Sect. 3 it follows that for $(x, \eta) \in a \times a$ with $x \neq \eta$ the set (2.2) is open in $[x, \eta]$ (and hence, by Remark 2.4 condition (i) of Proposition 2.2 holds) under one of the following conditions:

1. $\alpha = \alpha_1$, $f(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ is usc and compact-valued, and $c(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ has open graph.
2. $\alpha = \alpha_2$, $f(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ is lsc and $c(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ has closed graph.
3. $\alpha = \alpha_3$, $f(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ is usc and compact-valued, and $c(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ has closed graph.
4. $\alpha = \alpha_4$, $f(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ is lsc and $c(\cdot, \eta) : [x, \eta] \longrightarrow 2^Y$ has open graph.

We now introduce some generalized convexity and concavity notions for set-valued maps which are useful for checking condition (iii) of Proposition 2.1 with $\alpha = \alpha_i$, $i = 1, 2, 3, 4$.

Let $a \subset X$ be a nonempty convex set, $c' \subset Y$ be a convex cone, and $f : a \longrightarrow 2^Y$ be a set-valued map.

Map f is called c'-convex on a if for all $x_i \in a$, $i = 1, 2$, and $\gamma \in]0, 1[$

$$f(\gamma x_1 + (1 - \gamma)x_2) \subset \gamma f(x_1) + (1 - \gamma)f(x_2) - c'.$$

Map f is called proper c'-quasiconvex on a if for all $x_i \in a$, $i = 1, 2$, and $x \in \mathrm{co}\,\{x_i, i = 1, 2\}$,

$$\begin{aligned} \text{either} \quad & f(x) \subset f(x_1) - c', \\ \text{or} \quad & f(x) \subset f(x_2) - c'. \end{aligned}$$

Map f is called natural c'-quasiconvex on a if for all $x_i \in a$, $i = 1, 2$, and $\gamma \in]0, 1[$

$$f(\gamma x_1 + (1 - \gamma)x_2) \subset \mathrm{co}\,\{f(x_i), \quad i = 1, 2\} - c'.$$

Observe that the above notions of generalized convexity extend the corresponding notions of [12,40] for the single-valued case to the set-valued case. Observe also that convexity $\Rightarrow$ natural quasiconvexity, and proper quasiconvexity $\Rightarrow$ natural quasiconvexity. It is known [12] that even in single-valued case the converse of each of these implications is not true in general.

We now give some notions of generalized concavity.

Map f is called c'-concave on a if for all $x_i \in a$, $i = 1, 2$, and $\gamma \in]0, 1[$

$$\gamma f(x_1) + (1 - \gamma)f(x_2) \subset f(\gamma x_1 + (1 - \gamma)x_2) - c'.$$

Map f is called proper c'-quasiconcave on a if for all $x_i \in a$, $i = 1, 2$, and $x \in \operatorname{co}\{x_i,\ i = 1, 2\}$,

$$\begin{aligned} \text{either} \quad & f(x_1) \subset f(x) - c', \\ \text{or} \quad & f(x_2) \subset f(x) - c'. \end{aligned}$$

Map f is called generalized proper c'-quasiconcave on a if for all $x_i \in a$, $y_i \in f(x_i)$, $i = 1, 2$, and $x \in \operatorname{co}\{x_i,\ i = 1, 2\}$, there exists $y \in f(x)$ such that

$$\begin{aligned} \text{either} \quad & y_1 \in y - c', \\ \text{or} \quad & y_2 \in y - c'. \end{aligned}$$

Obviously, proper quasiconcavity $\Rightarrow$ generalized proper quasiconcavity, and the converse implication is true in case f being single-valued.

When f is single-valued, the above notions of concavity and proper quasiconcavity are reduced to the known notions in [12,40].

Let $h \subset Y$ be a nonempty set and let

$$\begin{aligned} Q_\alpha &= \{x \in a : \alpha(f(x), h + c')\}, \\ q_\alpha &= \{x \in a : \alpha(f(x), h + \operatorname{int} c')\}. \end{aligned}$$

When dealing with q_α we always assume that $\operatorname{int} c' \neq \emptyset$. The following proposition, whose easy proof is deleted, gives conditions under which Q_α and q_α are convex.

Proposition 2.3

1. *If f is generalized proper $(-c')$-quasiconcave on a, then Q_{α_1} and q_{α_1} are convex. If $\operatorname{int} c' \neq \emptyset$ and if f is generalized proper $(-\operatorname{int} c')$-quasiconcave on a, then for all $x \in Q_{\alpha_1}$, $\eta \in q_{\alpha_1}$ and $u \in]x, \eta[$ we have $u \in Q_{\alpha_1}$.*
2. *If f is natural $(-c')$-quasiconvex on a and if h is convex, then Q_{α_2} and q_{α_2} are convex. If in addition $\operatorname{int} c' \neq \emptyset$, then for all $x \in Q_{\alpha_2}$, $\eta \in q_{\alpha_2}$ and $u \in]x, \eta[$ we have $u \in q_{\alpha_2}$.*
3. *If f is c'-concave on a and if h is convex, then Q_{α_3} and q_{α_3} are convex. If in addition $\operatorname{int} c' \neq \emptyset$, then for all $x \in Q_{\alpha_3}$, $\eta \in q_{\alpha_3}$ and $u \in]x, \eta[$ we have $u \in q_{\alpha_3}$.*
4. *If f is proper c'-quasiconvex on a, then Q_{α_4} and q_{α_4} are convex. If $\operatorname{int} c' \neq \emptyset$ and f is proper $(\operatorname{int} c')$-quasiconvex on a, then for all $x \in Q_{\alpha_4}$, $\eta \in q_{\alpha_4}$ and $u \in]x, \eta[$ we have $u \in Q_{\alpha_4}$.*

3 Main result

From now on we assume that Y is a topological vector space, X and Z are locally convex Hausdorff topological vector spaces, and $K \subset X$ and $E \subset Z$ are nonempty convex sets. We also assume that $A : E \times K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $F : W \longrightarrow 2^Y$, $C : W \longrightarrow 2^Y$, $G : W \longrightarrow 2^Y$ and $D : W \longrightarrow 2^Y$ are set-valued maps with nonempty values, where $W := E \times K \times K \times K$ denotes the Cartesian product of topological spaces E, K, K and K.

In this section, we are interested in conditions under which there exists a solution of Problem (P_α) formulated in the Introduction. Before mentioning these conditions let us discuss some earlier results. In [41,Theorem 2] and in [6,Corollary 3.3] it is assumed that K is a compact convex set and $A(z, \xi) \equiv \widehat{A}(\xi)$ where $\widehat{A} : K \longrightarrow 2^K$ is a upper semicontinuous closed-valued map having open lower sections (see the Introduction). Under these assumptions $\widehat{A}$ must be a constant set-valued map, i.e., $\widehat{A}(\xi)$ does not depend on $\xi \in K$ (see [28,p. 178]). So Theorem 2 of [41] and Corollary 3.3 of [6] cannot be applied if $\widehat{A}$ is a non-constant set-valued map.

In [17,Theorem 3.1] the following existence result is obtained for Problem $(\widehat{P}_1)$ formulated in the Introduction: Assume that $E \subset Z$ and $K \subset X$ are nonempty compact convex sets. Assume, in addition, that

(i) $\widehat{A} : K \longrightarrow 2^K$ is a set-valued map with nonempty closed convex values and open lower sections.
(ii) $\widehat{B} : K \longrightarrow 2^E$ is a upper semicontinuous map with nonempty closed acyclic values.
(iii) $C_0 : K \longrightarrow 2^Y$ is a set-valued map with $\text{int}\, C_0(x) \neq \emptyset$ for all $x \in K$.
(iv) $\widehat{\varphi} : E \times K \times K \longrightarrow 2^Y$ is a set-valued map such that

(a) For any $\eta \in K$, the set

$$\{(z, x) \in E \times K : \widehat{\varphi}(z, x, \eta) \subset -\text{int}\, C_0(x)\}$$

is open in $E \times K$.

(b) For any $z \in E$, the set-valued map $\widehat{\varphi}(z, x, \eta)$ of the variable $(x, \eta) \in K \times K$ is weak Type II C_0-diagonally quasiconvex [17] in the variable η, i.e., for any finite set $L = \{\eta_i,\ i = 1, 2, ..., n\} \subset K$ and any point $\eta \in \text{co}\, L$ there exists a point $\eta_i \in L$ such that $\widehat{\varphi}(z, \eta, \eta_i) \not\subset -\text{int}\, C_0(\eta)$.

Under these assumptions Theorem 3.1 of [17] claims that there exists a solution of Problem $(\widehat{P}_1)$ with $\widehat{C}(x) = -\text{int}\, C_0(x)$ for all $x \in K$. We now give an example proving that these assumptions are not sufficient for the validity of the conclusion of Theorem 3.1 of [17].

Counterexample 3.1 Consider Problem $(\widehat{P}_1)$ where $X = Y = Z = \mathbb{R}$, $E = K = [0, 1] \subset \mathbb{R}$, $\widehat{B}(x) \equiv \{1\}$, $\widehat{C}(x) \equiv -\text{int}\, \mathbb{R}_+$, $\widehat{\varphi}(z, x, \eta) = \{z(x - \eta)\} \subset \mathbb{R}$ for all $z, x, \eta \in [0, 1]$ and

$$\widehat{A}(x) = \begin{cases} [0, 1], & \text{if } x \in [0, 1[\\ \{0\}, & \text{if } x = 1. \end{cases}$$

Let us set $C_0(x) \equiv \mathbb{R}_+$. To see that Theorem 3.1 of [17] can be applied to the above example, it suffices to verify condition (b) since other conditions are clear. Indeed, let us fix $z \in E = [0, 1]$. For any finite set $L = \{\eta_j,\ j = 1, 2, \ldots, n\} \subset K = [0, 1]$ and any point $\eta \in \text{co}\, L$, let us take $\eta_i = \min\{\eta_j,\ j = 1, 2, \ldots, n\}$. Then $\eta_i \leq \eta$ and hence, $z(\eta - \eta_i) \geq 0$ for all $z \in E = [0, 1]$. Thus, for fixed $z \in E$, $\widehat{\varphi}(z, \eta, \eta_i) \not\subset -\text{int}\, C_0(\eta)$, i.e., condition (b) is valid. By [17,Theorem 3.1] there exists a solution (z_0, x_0) of Problem $(\widehat{P}_1)$. Therefore, $x_0 \in [0, 1[$, $z_0 = 1$ and $z_0(x_0 - \eta) \geq 0$ for all $\eta \in \widehat{A}(x_0) = [0, 1]$, i.e., $x_0 \geq \eta$ for all $\eta \in [0, 1]$. This is impossible.

We have shown that Theorem 3.1 of [17] is incorrect. Example 3.1 also proves the incorrectness of Theorems 3.2 and 3.3 of [17]. Now, if in Example 3.1 we replace the condition $\widehat{C}(x) \equiv -\text{int}\, \mathbb{R}_+$ by the condition $\widehat{C}(x) \equiv \mathbb{R}_+$, then we will obtain a counterexample showing that Theorems 3.4 and 3.5 of [17] fail to hold. Thus, all the results of [17] giving the existence of solutions of Problems $(\widehat{P}_i)$, $i = 1, 2, 3, 4$, are incorrect.

In this section, we will obtain a general existence theorem for Problem (P_α) formulated in the Introduction. We will discuss in this section and in the subsequent section the special cases of Problem (P_α) where α is one of the relations $\alpha_i,\ i = 1, 2, 3, 4$. Observe that our main result (Theorem 3.1) can be applied to non-constant set valued map A, and can be used to derive correct existence theorems for Problems $(\widehat{P}_i)$, $i = 1, 2, 3, 4$, as well as some known results of [5,6,13,14,20,21,23,39,43].

The proof of the main result of this paper is based on two lemmas. Before formulating the first of them let us introduce the following notion of a β-pair where β is an arbitrary relation on 2^Y. The pair of maps (G, D) is called a β-pair if either (i) G is usc and compact-valued, and there exists a set-valued map $Q : W \longrightarrow 2^Y$ with open graph such that for all $w \in W$

$$\beta(G(w), D(w)) \Longleftrightarrow G(w) \not\subset Q(w). \tag{3.1}$$

or (ii) G is lsc, and there exists a set-valued map $Q : W \longrightarrow 2^Y$ with closed graph such that for all $w \in W$

$$\beta(G(w), D(w)) \Longleftrightarrow G(w) \subset Q(w). \tag{3.2}$$

Proposition 3.1

1. *If G is usc and compact-valued, and if D has open graph, then (G, D) is a β-pair with $\beta = \alpha_1$.*
2. *If G is lsc, and if D has closed graph, then (G, D) is a β-pair with $\beta = \alpha_2$.*
3. *If G is usc and compact-valued, and if D has closed graph, then (G, D) is a β-pair with $\beta = \alpha_3$.*
4. *If G is lsc, and if D has open graph, then (G, D) is a β-pair with $\beta = \alpha_4$.*

Proof To prove the first and second claims, it suffices to take $Q = D$. To prove the third and fourth claims, it is enough to take $Q = \widehat{D}$, where $\widehat{D}$ is defined by

$$\widehat{D}(w) = Y \setminus D(w) \quad (w \in W),$$

and observe that D has open (resp. closed) graph if and only if $\widehat{D}$ has closed (resp. open) graph. □

Lemma 3.1 *Let $A : E \times K \longrightarrow 2^K$ be a lsc map. Let β be an arbitrary relation on 2^Y. Let (G, D) be a β-pair. Then the map $\varphi_\beta : E \times K \longrightarrow 2^K$, defined by*

$$\varphi_\beta(z, \xi) = \{x \in K : \beta(G(z, \xi, x, \eta), D(z, \xi, x, \eta)), \quad \forall \eta \in A(z, \xi)\},$$

has a closed graph.

Proof To prove that the graph of φ_β, i.e., the set

$$\operatorname{gr} \varphi_\beta := \{(z, \xi, x) \in E \times K \times K : x \in \varphi_\beta(z, \xi)\},$$

is closed, it suffices to show that the complement of this graph in the topological space $E \times K \times K$ is open, i.e., if $(\widetilde{z}, \widetilde{\xi}, \widetilde{x}) \notin \operatorname{gr} \varphi_\beta$ then there exist neighbourhoods $U(\widetilde{z}), U(\widetilde{\xi})$ and $U(\widetilde{x})$ such that

$$(z, \xi, x) \notin \operatorname{gr} \varphi_\beta, \tag{3.3}$$

for all $(z, \xi, x) \in U(\widetilde{z}) \times U(\widetilde{\xi}) \times U(\widetilde{x})$. We first assume that the pair (G, D) satisfies condition (i) in the definition of a β-pair. Since $(\widetilde{z}, \widetilde{\xi}, \widetilde{x}) \notin \operatorname{gr} \varphi_\beta$ there exists $\widetilde{\eta} \in A(\widetilde{z}, \widetilde{\xi})$ such that $\overline{\beta}(G(\widetilde{w}), D(\widetilde{w}))$, i.e., $G(\widetilde{w}) \subset Q(\widetilde{w})$, where $\widetilde{w} := (\widetilde{z}, \widetilde{\xi}, \widetilde{x}, \widetilde{\eta})$ and Q is the map appearing in condition (i). In other words, we have

$$(\widetilde{w}, G(\widetilde{w})) \subset \operatorname{gr} Q. \tag{3.4}$$

Observe from [3,Proposition 7, p.73] that the map $g : W \longrightarrow 2^{W \times Y}$, defined by

$$g(w) = (w, G(w)), \tag{3.5}$$

is usc since G is usc and compact-valued. Since $g(\widetilde{w}) \subset \mathrm{gr}\, Q$ (see (3.4)) and since gr Q is an open set, we derive from the upper semicontinuity of g that there exist neighbourhoods $U_1(\widetilde{z})$, $U_1(\widetilde{\xi})$, $U(\widetilde{x})$ and $U(\widetilde{\eta})$ such that $g(w) \subset \mathrm{gr}\, Q$, i.e.,

$$G(w) \subset Q(w), \tag{3.6}$$

for all $w := (z, \xi, x, \eta) \in U(\widetilde{w}) := U_1(\widetilde{z}) \times U_1(\widetilde{\xi}) \times U(\widetilde{x}) \times U(\widetilde{\eta})$.

Observe now that $\widetilde{\eta} \in A(\widetilde{z}, \widetilde{\xi}) \cap U(\widetilde{\eta})$ which implies that $A(\widetilde{z}, \widetilde{\xi}) \cap U(\widetilde{\eta}) \neq \emptyset$. Therefore, by the lower semicontinuity of A there exist neighbourhoods $U(\widetilde{z}) \subset U_1(\widetilde{z})$ and $U(\widetilde{\xi}) \subset U_1(\widetilde{\xi})$ such that

$$A(z, \xi) \cap U(\widetilde{\eta}) \neq \emptyset, \tag{3.7}$$

for all $z \in U(\widetilde{z})$ and $\xi \in U(\widetilde{\xi})$. We claim that (3.3) holds for all $(z, \xi, x) \in U(\widetilde{z}) \times U(\widetilde{\xi}) \times U(\widetilde{x})$. Indeed, since $z \in U(\widetilde{z})$ and $\xi \in U(\widetilde{\xi})$, there exists $\eta \in A(z, \xi) \cap U(\widetilde{\eta})$ (see (3.7)). Setting $w = (z, \xi, x, \eta) \in U(\widetilde{w})$ we get from (3.6) $G(w) \subset Q(w)$, i.e.,

$$\overline{\beta}(G(w), D(w)) \quad (\text{see } (3.1)). \tag{3.8}$$

Thus, for all $(z, \xi, x) \in U(\widetilde{z}) \times U(\widetilde{\xi}) \times U(\widetilde{x})$ we find $\eta \in A(z, \xi)$ satisfying (3.8). This proves (3.3), as desired.

We now prove (3.3), assuming that the pair (G, D) satisfies condition (ii) in the definition of a β-pair. Since $(\widetilde{z}, \widetilde{\xi}, \widetilde{x}) \notin \mathrm{gr}\, \varphi_\beta$ there exists $\widetilde{\eta} \in A(\widetilde{z}, \widetilde{\xi})$ such that $\overline{\beta}(G(\widetilde{w}), D(\widetilde{w}))$, i.e., $G(\widetilde{w}) \not\subset Q(\widetilde{w})$ where $\widetilde{w} = (\widetilde{z}, \widetilde{\xi}, \widetilde{x}, \widetilde{\eta})$ and Q is the map appearing in condition (ii). In other words, for some $\widetilde{y} \in G(\widetilde{w})$ we have $\widetilde{y} \notin Q(\widetilde{w})$, or equivalently, $(\widetilde{w}, \widetilde{y}) \notin \mathrm{gr}\, Q$. From this and from the closedness of gr Q it follows that there exist neighbourhoods $U_1(\widetilde{z})$, $U_1(\widetilde{\xi})$, $U_1(\widetilde{x})$, $U_1(\widetilde{\eta})$ and $U_1(\widetilde{y})$ such that $(w, y) \notin \mathrm{gr}\, Q$, i.e.,

$$y \notin Q(w), \tag{3.9}$$

for all $w = (z, \xi, x, \eta) \in U_1(\widetilde{w}) := U_1(\widetilde{z}) \times U_1(\widetilde{\xi}) \times U_1(\widetilde{x}) \times U_1(\widetilde{\eta})$ and $y \in U_1(\widetilde{y})$. Observe that $G(\widetilde{w}) \cap U_1(\widetilde{y}) \neq \emptyset$ since $\widetilde{y} \in G(\widetilde{w}) \cap U_1(\widetilde{y})$. Hence, by the lower semicontinuity of G there exist neighbourhoods $U_2(\widetilde{z})$, $U_2(\widetilde{\xi})$, $U_2(\widetilde{x})$, and $U_2(\widetilde{\eta})$ such that

$$G(w) \cap U_1(\widetilde{y}) \neq \emptyset, \tag{3.10}$$

for all $w = (z, \xi, x, \eta) \in U_2(\widetilde{w}) := U_2(\widetilde{z}) \times U_2(\widetilde{\xi}) \times U_2(\widetilde{x}) \times U_2(\widetilde{\eta})$. Similarly, since $U_1(\widetilde{\eta}) \cap U_2(\widetilde{\eta})$ is an open set meeting $A(\widetilde{z}, \widetilde{\xi})$ at $\widetilde{\eta}$, and since A is lsc, there exist neighbourhoods $U_3(\widetilde{z})$ and $U(\widetilde{\xi}) \subset U_1(\widetilde{\xi}) \cap U_2(\widetilde{\xi})$ such that

$$A(z, \xi) \cap [U_1(\widetilde{\eta}) \cap U_2(\widetilde{\eta})] \neq \emptyset, \tag{3.11}$$

for all $z \in U_3(\widetilde{z})$ and $\xi \in U(\widetilde{\xi})$. Setting $U(\widetilde{z}) = U_1(\widetilde{z}) \cap U_2(\widetilde{z}) \cap U_3(\widetilde{z})$, $U(\widetilde{x}) = U_1(\widetilde{x}) \cap U_2(\widetilde{x})$ we infer that (3.3) holds for all $(z, \xi, x) \in U(\widetilde{z}) \times U(\widetilde{\xi}) \times U(\widetilde{x})$. Indeed, since $z \in U(\widetilde{z})$ and $\xi \in U(\widetilde{\xi})$ there exists $\eta \in A(z, \xi) \cap [U_1(\widetilde{\eta}) \cap U_2(\widetilde{\eta})]$ (see (3.11)). Since $w = (z, \xi, x, \eta) \in U_2(\widetilde{w})$ there exists $y \in U_1(\widetilde{y})$ with $y \in G(w)$ (see (3.10)). Therefore, (3.9) holds. Since $y \in G(w)$ this implies that $G(w) \not\subset Q(w)$ which, by (3.2), is equivalent to condition (3.8). Thus, for all $(z, \xi, x) \in U(\widetilde{z}) \times U(\widetilde{\xi}) \times U(\widetilde{x})$ we find $\eta \in A(z, \xi)$ satisfying (3.8). This proves (3.3), as desired.

It is worth noticing that the proof of Lemma 3.1 does not require the convexity of the sets E and K. □

Corollary 3.1 *Let (G, D) be a β-pair and let K' be a nonempty subset of K. Then for all $(z, \xi) \in E \times K$ and $\eta' \in K'$, the set*

$$\{x \in K' : \beta(G(z, \xi, x, \eta'), D(z, \xi, x, \eta'))\}$$

is closed in K'.

Proof Let us fix $\eta' \in K'$ and consider the constant map

$$(z, \xi) \in E \times K \mapsto A'(z, \xi) = \{\eta'\}.$$

Then by Lemma 3.1 the map $\varphi'_\beta : E \times K \longrightarrow 2^K$, defined by

$$\begin{aligned}\varphi'_\beta(z, \xi) &= \{x \in K : \beta(G(z, \xi, x, \eta), D(z, \xi, x, \eta)), \quad \forall \eta \in A'(z, \xi)\}\\ &= \{x \in K : \beta(G(z, \xi, x, \eta'), D(z, \xi, x, \eta'))\}\end{aligned}$$

is closed. Hence, for all $(z, \xi) \in E \times K$ $\varphi'_\beta(z, \xi)$ is closed in K. Since K' is a subset of K, it follows that $K' \cap \varphi'_\beta(z, \xi)$ is closed in K'. □

Let $\widetilde{G} : K \longrightarrow 2^Y$ and $\widetilde{D} : K \longrightarrow 2^Y$ be set-valued maps with nonempty values. Let

$$K_\alpha = \{x \in K : \alpha(\widetilde{G}(x), \widetilde{D}(x))\}.$$

Corollary 3.2 *The set K_α is closed in K and the set $K_{\overline{\alpha}}$ is open in K if one of the following conditions holds:*

1. *$\alpha = \alpha_1$, $\widetilde{G}$ is usc and compact-valued, and $\widetilde{D}$ has open graph.*
2. *$\alpha = \alpha_2$, $\widetilde{G}$ is lsc, and $\widetilde{D}$ has closed graph.*
3. *$\alpha = \alpha_3$, $\widetilde{G}$ is usc and compact-valued, and $\widetilde{D}$ has closed graph.*
4. *$\alpha = \alpha_4$, $\widetilde{G}$ is lsc, and $\widetilde{D}$ has open graph.*

Proof Since $K_{\overline{\alpha}}$ is the complement of K_α in K, it suffices to prove that K_α is closed in K. Setting $\beta = \alpha$, $G(z, \xi, x, \eta) \equiv \widetilde{G}(x)$ and $D(z, \xi, x, \eta) \equiv \widetilde{D}(x)$, we see that φ_β is a constant (set-valued) map. Namely, for all $(z, \xi) \in E \times K$, $\varphi_\beta(z, \xi) \equiv K_\beta$. Therefore, the closedness (in K) of K_α is a consequence of Proposition 3.1 and Lemma 3.1 applied to the pair $(\widetilde{G}, \widetilde{D})$. □

We now formulate the second lemma which is needed for the proof of the main result of this paper.

Lemma 3.2 (see [37]) *Let $A : E \times K \longrightarrow 2^K$ be a compact upper semicontinuous map with closed values, $B : E \times K \longrightarrow 2^E$ be a compact acyclic map, and $\varphi : E \times K \longrightarrow 2^K$ be a closed map such that, for all $(z, \xi) \in E \times K$, the set $\psi(z, \xi) = \varphi(z, \xi) \cap A(z, \xi)$ is nonempty and acyclic. Then the map $V := B \times \psi$ has a fixed point.*

Proof This is a consequence of Theorem 2.1 applied to map V. □

Given set-valued maps F, C, G and D from $W := E \times K \times K \times K$ to Y, let us set

$$\begin{aligned}T_\alpha(z, \xi) &= \{x \in A(z, \xi) : \alpha(F(z, \xi, x, \eta), C(z, \xi, x, \eta)), \quad \forall \eta \in A(z, \xi)\},\\ S_\beta(z, \xi) &= \{x \in A(z, \xi) : \beta(G(z, \xi, x, \eta), D(z, \xi, x, \eta)), \quad \forall \eta \in A(z, \xi)\}.\end{aligned}$$

The main result of this paper is expressed in the following theorem.

Theorem 3.1 *Let $A : E \times K \longrightarrow 2^K$ be a compact continuous map with closed values, and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Let α and β be arbitrary relations on 2^Y. Let (G, D) be a β-pair such that*

(i) $S_\beta \subset T_\alpha$.
(ii) *For all* $(z, \xi) \in E \times K$, $S_\beta(z, \xi)$ *is nonempty and acyclic.*

Then there exists a solution of Problem (P_α).

Proof Consider the map $\tau_\alpha : E \times K \longrightarrow 2^{E\times K}$ defined by $\tau_\alpha(z, \xi) = B(z, \xi) \times T_\alpha(z, \xi)$ for all $(z, \xi) \in E \times K$. Observe that Problem (P_α) has a solution $(z_0, x_0) \in E \times K$ if and only if (z_0, x_0) is a fixed point of map τ_α. On the other hand, by condition (i) $V_\beta \subset \tau_\alpha$ where $V_\beta := B \times S_\beta$. Hence, the problem of finding a solution of (P_α) reduces to that of finding a fixed point of V_β. Observe that

$$S_\beta(z, \xi) = \varphi_\beta(z, \xi) \cap A(z, \xi)$$

where φ_β, defined in Lemma 3.1, is a closed map. Applying Lemma 3.2 with $\psi = S_\beta$ and $\varphi = \varphi_\beta$, we derive that V_β has a fixed point, as desired. □

Theorem 3.2 *Let* $A : E \times K \longrightarrow 2^K$ *be a compact continuous map with closed values, and* $B : E \times K \longrightarrow 2^E$ *be a compact acyclic map. Let* α *be an arbitrary relation on* 2^Y. *Let* (F, C) *be a* α*-pair such that, for all* $(z, \xi) \in E \times K$, $T_\alpha(z, \xi)$ *is nonempty and acyclic. Then there exists a solution of Problem* (P_α).

Proof This is a consequence of Theorem 3.1 with $\beta = \alpha$ and $(G, D) = (F, C)$. □

Corollary 3.3 *Let A and B be as in Theorem 3.2. Let* $f : E \times K \times K \longrightarrow Y$ *be a single-valued continuous map and* $C' : E \times K \longrightarrow 2^Y$ *be a set-valued map such that, for all* $(z, \xi) \in E \times K$, $C'(z, \xi) \neq Y$ *and* $C'(z, \xi)$ *is a closed convex cone with nonempty interior. Assume additionally that*

(i) *The map*

$$(z, \xi) \in E \times K \longmapsto \operatorname{int} C'(z, \xi)$$

has open graph.

(ii) *For all* $(z, \xi) \in E \times K$, *the set*

$$\{x \in A(z, \xi) : [f(z, \xi, x) - f(z, \xi, A(z, \xi))] \cap \operatorname{int} C'(z, \xi) = \emptyset\} \tag{3.12}$$

is acyclic.

Then there exists $(z_0, x_0) \in E \times K$ *such that* $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ *and, for all* $\eta \in A(z_0, x_0)$,

$$f(z_0, x_0, x_0) - f(z_0, x_0, \eta) \notin \operatorname{int} C'(z_0, x_0).$$

Proof For all $w = (z, \xi, x, \eta) \in W := E \times K \times K \times K$, let us set $F(w) = f(z, \xi, \eta)$, $C(w) = f(z, \xi, x) - \operatorname{int} C'(z, \xi)$. (Thus, F does not depend on x and C does not depend on η.) Since f is a single-valued continuous map, we see that F is continuous, and C has open graph. By Proposition 3.1 (F, C) is a α-pair with $\alpha = \alpha_1$. On the other hand, the set (3.12) is exactly the set $T_{\alpha_1}(z, \xi)$ which is acyclic by assumption (ii). In addition, this set is nonempty (see [18,31]). To complete the proof it remains to apply Theorem 3.2 with $\alpha = \alpha_1$. □

Remark 3.1 Theorem 1 of [20], Theorem 2.1 of [21], Theorem 3 of [23], Theorem 3.1 of [5] and Theorem 3.8 of [6] are direct consequences of Corollary 3.3.

Corollary 3.4 *Let A and B be as in Theorem 3.2. Let $F : W \longrightarrow 2^Y$ be lsc and let $C : W \longrightarrow 2^Y$ have closed graph. Assume that, for all $(z, \xi) \in E \times K$, the set*

$$\{x \in A(z, \xi) : F(z, \xi, x, \eta) \subset C(z, \xi, x, \eta), \quad \forall \eta \in A(z, \xi)\}$$

is nonempty and acyclic. Then there exists a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ and for all $\eta \in A(z_0, x_0)$

$$F(z_0, x_0, x_0, \eta) \subset C(z_0, x_0, x_0, \eta).$$

Proof Apply Theorem 3.2 with $\alpha = \alpha_2$ and observe by Proposition 3.1 that (F, C) is a α_2-pair. □

Remark 3.2 Corollary 3.4 improves Theorem 3.1 of [43] and proves that both Problems (P_1) and (P_2) mentioned in [43] (i.e. Problems $(\widetilde{P}_1)$ and $(\widetilde{P}_2)$ formulated in the Introduction) can be investigated by a unified approach. This remark is based on the fact that Problem $(\widetilde{P}_1)$ (resp. $(\widetilde{P}_2)$) corresponds to the case when $F(z, \xi, x, \eta)$ does not depend on x (resp. η) and $C(z, \xi, x, \eta)$ does not depend on η (resp. x).

Remark 3.3 With the help of Corollary 3.4 we can derive Theorem 1 of [13], and Theorems 3.1 and 3.2 of [39] under conditions which are much weaker than those of [13,39]. For a detailed discussion, see [43].

Corollary 3.5 *Let Y_i, $i = 1, 2$, be topological vector spaces. Let $E \subset Z$ and $K_i \subset X_i$, $i = 1, 2$, be nonempty convex sets where Z and X_i are locally convex Hausdorff topological vector spaces. Let $B' : E \times K_1 \times K_2 \longrightarrow 2^E$ be a compact acyclic map and let $A_i : E \times K_1 \times K_2 \longrightarrow 2^{K_i}$, $i = 1, 2$, be compact continuous maps with nonempty closed values. Let $C_i : E \times K_1 \times K_2 \longrightarrow 2^{Y_i}$, $i = 1, 2$, be set-valued maps such that, for each $(z, \xi_1, \xi_2) \in E \times K_1 \times K_2$ and each $i = 1, 2$, $C_i(z, \xi_1, \xi_2)$ is a closed convex cone with nonempty interior and $C_i(z, \xi_1, \xi_2) \neq Y_i$. Assume that the set-valued maps*

$$(z, \xi_1, \xi_2) \in E \times K_1 \times K_2 \longmapsto \operatorname{int} C_i(z, \xi_1, \xi_2), \quad i = 1, 2,$$

have open graphs. Assume that $\varphi_i : E \times K_1 \times K_2 \longrightarrow Y_i$, $i = 1, 2$, are single-valued continuous maps such that at least one of the following conditions (i) and (ii) is satisfied:

(i) For each $(z, \xi_1, \xi_2) \in E \times K_1 \times K_2$, the sets

$$a_1(z, \xi_1, \xi_2) = \{x_1 \in A_1(z, \xi_1, \xi_2) : \\ [\varphi_1(z, x_1, \xi_2) - \varphi_1(z, A_1(z, \xi_1, \xi_2), \xi_2)] \cap \operatorname{int} C_1(z, \xi_1, \xi_2) = \emptyset\},$$

$$a_2(z, \xi_1, \xi_2) = \{x_2 \in A_2(z, \xi_1, \xi_2) : \\ [\varphi_2(z, \xi_1, x_2) - \varphi_2(z, \xi_1, A_2(z, \xi_1, \xi_2))] \cap \operatorname{int} C_2(z, \xi_1, \xi_2) = \emptyset\},$$

are acyclic.

(ii) For each $(z, \xi_1, \xi_2) \in E \times K_1 \times K_2$, $A_i(z, \xi_1, \xi_2)$, $i = 1, 2$, are convex, $\varphi_1(z, \cdot, \xi_2)$ is properly $C_1(z, \xi_1, \xi_2)$-quasiconvex and $\varphi_2(z, \xi_1, \cdot)$ is properly $C_2(z, \xi_1, \xi_2)$-quasiconvex.

Then there exists a point $(z_0, x_1^0, x_2^0) \in E \times K_1 \times K_2$ such that $z_0 \in B'(z_0, x_1^0, x_2^0)$, $x_i^0 \in A_i(z_0, x_1^0, x_2^0)$, $i = 1, 2$, and

$$\varphi_1(z_0, x_1^0, x_2^0) - \varphi_1(z_0, \eta_1, x_2^0) \notin \operatorname{int} C_1(z_0, x_1^0, x_2^0), \quad \forall \eta_1 \in A_1(z_0, x_1^0, x_2^0),$$
$$\varphi_2(z_0, x_1^0, x_2^0) - \varphi_2(z_0, x_1^0, \eta_2) \notin \operatorname{int} C_2(z_0, x_1^0, x_2^0), \quad \forall \eta_2 \in A_2(z_0, x_1^0, x_2^0).$$

Proof Let us set $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ and $K = K_1 \times K_2$. For $z \in E$, $\xi = (\xi_1, \xi_2) \in K = K_1 \times K_2$, $x = (x_1, x_2) \in K = K_1 \times K_2$ and $\eta = (\eta_1, \eta_2) \in K = K_1 \times K_2$ let us set

$$\begin{aligned}
F(z, \xi, x, \eta) &= \big\{[\varphi_1(z, x_1, \xi_2) - \varphi_1(z, \eta_1, \xi_2)] \times [\varphi_2(z, \xi_1, x_2) - \varphi_2(z, \xi_1, \eta_2)]\big\} \subset Y = Y_1 \times Y_2,\\
A(z, \xi) &= A_1(z, \xi_1, \xi_2) \times A_2(z, \xi_1, \xi_2) \subset K = K_1 \times K_2,\\
B(z, \xi) &= B'(z, \xi_1, \xi_2) \subset E,\\
C(z, \xi, x, \eta) &= [Y_1 \backslash \operatorname{int} C_1(z, \xi_1, \xi_2)] \times [Y_2 \backslash \operatorname{int} C_2(z, \xi_1, \xi_2)] \subset Y = Y_1 \times Y_2.
\end{aligned}$$

To apply Corollary 3.4 we need to prove that, for each $z \in E$ and each $\xi = (\xi_1, \xi_2) \in K_1 \times K_2 = K$, the set

$$\widetilde{A}(z, \xi) := \{x \in A(z, \xi) : F(z, \xi, x, \eta) \subset C(z, \xi, x, \eta), \quad \forall \eta \in A(z, \xi)\}$$

is nonempty and acyclic. Indeed, by the above definitions of F and A we can verify that

$$\widetilde{A}(z, \xi) = \widetilde{a}_1(z, \xi) \times \widetilde{a}_2(z, \xi) \subset K_1 \times K_2 \subset X_1 \times X_2$$

where

$$\begin{aligned}
\widetilde{a}_1(z, \xi) &= \{x_1 \in A_1(z, \xi_1, \xi_2) :\\
&\qquad \varphi_1(z, x_1, \xi_2) - \varphi_1(z, \eta_1, \xi_2) \in Y_1 \backslash \operatorname{int} C_1(z, \xi_1, \xi_2), \quad \forall \eta_1 \in A_1(z, \xi_1, \xi_2)\}\\
&= a_1(z, \xi_1, \xi_2),\\
\widetilde{a}_2(z, \xi) &= \{x_2 \in A_2(z, \xi_1, \xi_2) :\\
&\qquad \varphi_2(z, \xi_1, x_2) - \varphi_2(z, \xi_1, \eta_2) \in Y_2 \backslash \operatorname{int} C_2(z, \xi_1, \xi_2), \quad \forall \eta_2 \in A_2(z, \xi_1, \xi_2)\}\\
&= a_2(z, \xi_1, \xi_2).
\end{aligned}$$

(For the definition of sets $a_i(z, \xi_1, \xi_2)$, $i = 1, 2$, see condition (i) of Corollary 3.5.) Since $\varphi_1(z, A_1(z, \xi_1, \xi_2), \xi_2)$ and $\varphi_2(z, \xi_1, A_2(z, \xi_1, \xi_2))$ are compact, it follows from [18,31] that $a_i(z, \xi_1, \xi_2) \neq \emptyset$, $i = 1, 2$. Therefore, $\widetilde{A}(z, \xi)$ is nonempty.

If condition (i) holds, then $\widetilde{A}(z, \xi)$ is acyclic since it is the product of acyclic sets $\widetilde{a}_i(z, \xi)$, $i = 1, 2$. We now prove that $\widetilde{A}(z, \xi)$ is acyclic if condition (ii) of Corollary 3.5 holds. Indeed, observe that

$$\widetilde{a}_1(z, \xi) = A_1(z, \xi_1, \xi_2) \cap a_1'(z, \xi_1, \xi_2),$$

where $\xi = (\xi_1, \xi_2)$ and the set

$$\begin{aligned}
a_1'(z, \xi_1, \xi_2) := \big\{&x_1 \in K_1 :\\
&\{\varphi_1(z, x_1, \xi_2)\} \cap [\varphi_1(z, A_1(z, \xi_1, \xi_2), \xi_2) + \operatorname{int} C_1(z, \xi_1, \xi_2)] = \emptyset\big\}
\end{aligned}$$

is convex by the fourth claim of Proposition 2.3. Therefore, $\widetilde{a}_1(z, \xi)$ being the intersection of two convex sets is convex. By a similar argument we can verify that $\widetilde{a}_2(z, \xi)$ is convex. Therefore, $\widetilde{A}(z, \xi)$ being the product of two convex sets is convex and hence, it is acyclic, as desired.

Now, applying Corollary 3.4 we claim that there exist points $z_0 \in E$, $x_0 := (x_1^0, x_2^0) \in K = K_1 \times K_2$ such that $z_0 \in B(z_0, x_0) = B'(z_0, x_1^0, x_2^0)$, $x_0 = (x_1^0, x_2^0) \in A(z_0, x_0) = A_1(z_0, x_1^0, x_2^0) \times A_2(z_0, x_1^0, x_2^0)$ and

$$\begin{aligned}
F(z_0, x_0, x_0, \eta) &= \{[\varphi_1(z_0, x_1^0, x_2^0) - \varphi_1(z_0, \eta_1, x_2^0)] \times [\varphi_2(z_0, x_1^0, x_2^0) - \varphi_2(z_0, x_1^0, \eta_2)]\}\\
&\subset [Y_1 \backslash \operatorname{int} C_1(z_0, x_1^0, x_2^0)] \times [Y_2 \backslash \operatorname{int} C_2(z_0, x_1^0, x_2^0)]
\end{aligned}$$

for all $\eta = (\eta_1, \eta_2) \in A(z_0, x_0) = A_1(z_0, x_1^0, x_2^0) \times A_2(z_0, x_1^0, x_2^0)$. This proves that the point (z_0, x_1^0, x_2^0) satisfies the conclusion of Corollary 3.5. □

Remark 3.4 When E and B are absent, $\varphi_i(z, \xi_1, \xi_2)$ and $A_i(z, \xi_1, \xi_2)$, $i = 1, 2$, do not depend on z, and $C_i(z, \xi_1, \xi_2)$, $i = 1, 2$, do not depend on z, ξ_1 and ξ_2, Corollary 3.5 becomes a theorem in [14,p. 710] dealing with the existence of solutions of symmetric quasi-equilibrium problems.

Remark 3.5 Corollary 3.3 can be derived from Corollary 3.5 by setting $X_1{=}X_2{=}X$, $Y_1{=}Y_2 = Y$, $K_1 = K_2 = K$, $A_1(z, \xi_1, \xi_2) = \{\xi_2\}$, $A_2(z, \xi_1, \xi_2) = A(z, \xi_1)$, $B'(z, \xi_1, \xi_2) = B(z, \xi_1)$, $C_1(z, \xi_1, \xi_2) = C_2(z, \xi_1, \xi_2) = C'(z, \xi_1)$ and $\varphi_1(z, \xi_1, \xi_2) = \varphi_2(z, \xi_1, \xi_2) = f(z, \xi_1, \xi_2)$, where $z \in E, \xi_1 \in K$, $\xi_2 \in K$, and A, B, C' and f are set-valued maps mentioned in Corollary 3.3.

Now, for each fixed point $(z, \xi) \in E \times K$ we set $a = A(z, \xi)$ and we consider the set-valued map $f_{z,\xi} : a \times a \longrightarrow 2^Y$ defined by setting $f_{z,\xi}(x, \eta) = F(z, \xi, x, \eta)$ for $(x, \eta) \in a \times a$. The maps $c_{z,\xi}$, $g_{z,\xi}$ and $d_{z,\xi}$ are defined similarly.

Theorem 3.3 *Let $A : E \times K \longrightarrow 2^K$ be a compact continuous map with closed convex values, and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Let α be an arbitrary relation on 2^Y. Let (F, C) be a α-pair such that, for all $(z, \xi) \in E \times K$, $T_\alpha(z, \xi)$ is acyclic and, in addition, the following conditions are satisfied :*

(i) For all $x \in A(z, \xi)$, the set

$$\{\eta \in A(z, \xi) : \overline{\alpha}(f_{z,\xi}(x, \eta), c_{z,\xi}(x, \eta))\}$$

is convex.

(ii) For all $x \in A(z, \xi)$, $\alpha(f_{z,\xi}(x, x), c_{z,\xi}(x, x))$.

Then there exists a solution of Problem (P_α).

Proof First observe from Corollary 3.1 with $K' = A(z, \xi)$ that for all $\eta \in A(z, \xi)$ the set $\{x \in A(z, \xi) : \alpha(f_{z,\xi}(x, \eta), c_{z,\xi}(x, \eta))\}$ is closed in $A(z, \xi)$. Now making use of Proposition 2.1 with $\beta = \alpha$, $f = g = f_{z,\xi}$, $c = d = c_{z,\xi}$ we derive that $T_\alpha(z, \xi)$ is nonempty for all $(z, \xi) \in E \times K$. It remains to apply Theorem 3.2 to obtain the conclusion of Theorem 3.3. □

Remark 3.6 Proposition 2.3 can be used to derive condition (i) of Theorem 3.3 if $C(z, \xi, x, \eta)$ does not depend on η and is of the form

$$C(z, \xi, x, \eta) = H(z, \xi, x) + \operatorname{int} C'(z, \xi, x) \tag{3.13}$$

or

$$C(z, \xi, x, \eta) = H(z, \xi, x) + C'(z, \xi, x) \tag{3.14}$$

where $H(z, \xi, x) \subset Y$ is a nonempty set and $C'(z, \xi, x) \subset Y$ is a nonempty convex cone. For example, if C is of the form (3.13) with $H(z, \xi, x)$ being convex and if $f_{z,\xi}(x, \cdot)$ is natural $[-C'(z, \xi, x)]$-quasiconvex on $A(z, \xi)$, then condition (i) of Theorem 3.3 holds for $\alpha = \alpha_1$.

From Remark 3.6 we see that combining Theorem 3.3, and Propositions 3.1 and 2.3 will give existence results for Problem (P_{α_i}) where C is of the form (3.13) or (3.14). We will not formulate all of these existence results. But, as an illustrating example, we restrict ourselves to an existence result for Problem (P_{α_1}).

Corollary 3.6 *Let A and B be as in Theorem 3.3. Assume that $F : W \longrightarrow 2^Y$ is usc and compact-valued, and $C : W \longrightarrow 2^Y$ is of the form (3.13) where $H(z, \xi, x) \subset Y$ is a nonempty convex set and $C'(z, \xi, x) \subset Y$ is a convex cone with nonempty interior. Assume additionally that*

(i) The map C defined by (3.13) has open graph.
(ii) For all $(z,\xi) \in E \times K$ and $x \in A(z,\xi)$, $F(z,\xi,x,\cdot)$ is natural $[-C'(z,\xi,x)]$-quasiconvex on $A(z,\xi)$.
(iii) For all $(z,\xi) \in E \times K$ and $x \in A(z,\xi)$,

$$F(z,\xi,x,x) \not\subset H(z,\xi,x) + \mathit{int}\, C'(z,\xi,x).$$

If for all $(z,\xi) \in E \times K$ the set $T_{\alpha_1}(z,\xi)$, i.e., the set

$$\{x \in A(z,\xi) : F(z,\xi,x,\eta) \not\subset H(z,\xi,x) + \mathit{int}\, C'(z,\xi,x), \quad \forall \eta \in A(z,\xi)\},$$

is acyclic, then there exists a solution of Problem (P_{α_1}).

Proof Use Theorem 3.3 and Remark 3.6 with $\alpha = \alpha_1$. □

Remark 3.7 The requirement of Corollary 3.6 that the set $T_{\alpha_1}(z,\xi)$ is acyclic is automatically satisfied if $H(z,\xi,x)$ and $C'(z,\xi,x)$ do not depend on x, i.e., $H(z,\xi,x) \equiv H(z,\xi)$ and $C'(z,\xi,x) \equiv C'(z,\xi)$, and if $F(z,\xi,\cdot,\eta)$ is generalized proper $[-C'(z,\xi)]$-quasiconcave on $A(z,\xi)$. This is because in this case $T_{\alpha_1}(z,\xi)$ is convex (Proposition 2.3).

Theorem 3.4 *Let $A : E \times K \longrightarrow 2^K$ be a compact continuous map with closed convex values, and $B : E \times K \longrightarrow 2^E$ be a compact acyclic map. Let α and β be arbitrary relations on 2^Y. Let (G, D) be a β-pair such that, for all $(z,\xi) \in E \times K$, $S_\beta(z,\xi)$ is acyclic. Assume additionally that, for each $(z,\xi) \in E \times K$, the set $a = A(z,\xi)$ and the maps $f = f_{z,\xi}$, $c = c_{z,\xi}$, $g = g_{z,\xi}$ and $d = d_{z,\xi}$ satisfy conditions (i), (iii), (iv) of Proposition 2.1 and conditions (i), (ii) and (iii) of Proposition 2.2. Then there exists a solution of Problem (P_α).*

Proof This is a consequence of Theorem 3.1 since by Proposition 2.1 $S_\beta(z,\xi)$ is nonempty, and by Proposition 2.2 $S_\beta(z,\xi) \subset T_\alpha(z,\xi)$ for all $(z,\xi) \in E \times K$. (Condition (ii) of Proposition 2.1 is satisfied by Corollary 3.1.) □

The rest of this section is devoted to discussions related to the sets $S_\beta(z,\xi)$ and $T_\alpha(z,\xi)$. In some existence results we have assumed that $S_\beta(z,\xi)$ or $T_\alpha(z,\xi)$ is acyclic. Such an assumption for sets similar to $S_\beta(z,\xi)$ and $T_\alpha(z,\xi)$ is used in several papers dealing with vector equilibrium problems (see e.g. [20,21,27,29,42] and references therein). Observe that checking this assumption becomes easier if some convexity properties are introduced (see [20,21,27,29,42]). In case of $S_\beta(z,\xi)$ we have the following result which gives sufficient conditions for $S_\beta(z,\xi)$ to be acyclic. (Similar result can be formulated for $T_\alpha(z,\xi)$.)

Proposition 3.2 *Let (z,ξ) be a fixed point of $E \times K$. Assume that $A(z,\xi)$ is convex and, for each $\eta \in K$, the set*

$$\{x \in K : \beta(G(z,\xi,x,\eta), D(z,\xi,x,\eta))\} \tag{3.15}$$

is convex. Then $S_\beta(z,\xi)$ is convex (and hence, it is acyclic).

Proof Let us denote the convex set (3.15) by $S_\beta(z,\xi,\eta)$. Then

$$S'_\beta(z,\xi) := \bigcap_{\eta \in A(z,\xi)} S_\beta(z,\xi,\eta)$$

is convex since it is expressed as the intersection of a family of convex sets. On the other hand, by the definition of $S_\beta(z,\xi)$ we get

$$S_\beta(z,\xi) = A(z,\xi) \cap S'_\beta(z,\xi).$$

This proves the convexity of $S_\beta(z,\xi)$ since both sets $A(z,\xi)$ and $S'_\beta(z,\xi)$ are convex. □

Remark 3.8 If $D(z, \xi, x, \eta)$ does not depend on x and can be expressed as the sum of a convex set and a convex cone, then the convexity of the set (3.15), with $\beta = \alpha_1, \alpha_2, \alpha_3, \alpha_4$, is assured by an appropriate convexity property of the map $G(z, \xi, \cdot, \eta)$ (see the conditions for convexity of the set Q_{α_i} in Proposition 2.3).

Remark 3.9 For each $(z, \xi) \in E \times K$ the nonemptiness of $S_\beta(z, \xi)$ can be derived from Proposition 2.1 with suitable assumptions of $a = A(z, \xi)$, $g(x, \eta) = G(z, \xi, x, \eta)$ and $d(x, \eta) = D(z, \xi, x, \eta)$ (see the proof of Theorem 3.4). Similarly, the nonemptiness of $T_\alpha(z, \xi)$ can be obtained from Proposition 2.1 (see the proof of Theorem 3.3).

Remark 3.10 We have seen from the above discussions that by means of Proposition 3.2 (resp. Proposition 2.1) sufficient conditions for the set $S_\beta(z, \xi)$ to be acyclic (resp. nonempty) can be given in terms of conditions imposed on the maps F, G, C, D and A. Thus, these conditions guarantee the validity of the assumption (ii) of Theorem 3.1. On the other hand, the assumption (i) of Theorem 3.1 is usually assured by a pseudomonotonicity type condition where some links between F, G, C, D and A are required to be satisfied. So, with the help of Propositions 3.2 and 2.1, and a pseudomonotonicity type assumption, the formulation of the assumptions (i) and (ii) in Theorem 3.1 can be replaced by suitable conditions imposed on F, G, C, D and A. In other words, we can derive from Theorem 3.1 sufficient conditions for the existence of solutions of Problem (P_α) without mentioning the sets $S_\beta(z, \xi)$ and $T_\alpha(z, \xi)$. (Similar remarks can be made for Theorems 3.2–3.4.) For the case $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4$, existence theorems for Problem (P_α) will be obtained in Sect. 4 without the direct appearance of these sets.

4 Some special cases

Throughout this section we assume that $A : E \times K \longrightarrow 2^K$ is a compact continuous map with closed convex values and $B : E \times K \longrightarrow 2^E$ is a compact acyclic map. We assume that for all $(z, \xi, x) \in E \times K \times K$ $H(z, \xi, x) \subset Y$ is a nonempty set and $C'(z, \xi, x) \subset Y$ is a convex cone with nonempty interior. In Theorems 4.1 and 4.4 we additionally assume that, for all $(z, \xi, x) \in E \times K \times K$, $H(z, \xi, x)$ is a convex set. This convexity property of $H(z, \xi, x)$ is not needed for the validity of Theorems 4.2 and 4.3.

This section is devoted to existence theorems for Problem (P_α) where $C(z, \xi, x, \eta)$ does not depend on η and is of the form (3.13) (for $\alpha = \alpha_1$ and $\alpha = \alpha_4$) or the form (3.14) (for $\alpha = \alpha_2$ and $\alpha = \alpha_3$). These are Theorems 4.1–4.4, all of which are consequences of Theorem 3.4, Corollary 3.2 and Propositions 3.1 and 2.3. Condition (i) in these theorems plays the role of a pseudomonotonicity assumption and is inspired by the corresponding pseudomonotonicity condition (i) of Theorems 1 and 1A of [35]. We will give a detailed proof of Theorem 4.1 and delete the similar proof of Theorems 4.2–4.4.

Theorem 4.1 *Assume that $F : W \longrightarrow 2^Y$ is lsc map. Assume additionally that*

(i) For all $(z, \xi) \in E \times K$ and $(x, \eta) \in A(z, \xi) \times A(z, \xi)$,

$$F(z, \xi, x, \eta) \not\subset H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)$$
$$\Longrightarrow F(z, \xi, \eta, x) \subset H(z, \xi, \eta) + C'(z, \xi, \eta).$$

(ii) For all $(z, \xi) \in E \times K$ and $(x, \eta) \in A(z, \xi) \times A(z, \xi)$ with $x \neq \eta$, if $F(z, \xi, x, \eta) \subset H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)$ then $F(z, \xi, u, \eta) \subset H(z, \xi, u) + \operatorname{int} C'(z, \xi, u)$ for some $u \in]x, \eta[$.

(iii) *For all* $(z, \xi, x) \in E \times K \times K$, $F(z, \xi, x, \cdot)$ *is natural* $[-C'(z, \xi, x)]$*-quasiconvex on* $A(z, \xi)$.

(iv) *For all* $(z, \xi) \in E \times K$ *and* $x \in A(z, \xi)$,

$$F(z, \xi, x, x) \not\subset H(z, \xi, x) + \operatorname{int} C'(z, \xi, x).$$

(v) *The map*

$$(z, \xi, \eta) \in E \times K \times K \longmapsto H(z, \xi, \eta) + C'(z, \xi, \eta)$$

has closed graph.

Then there exists a solution of Problem (P_{α_1}) *with* C *being defined by* (3.13).

Proof This is a consequence of Theorem 3.4 with $\alpha = \alpha_1$, $\beta = \alpha_2$, $C(z, \xi, x, \eta) = H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)$, $G(z, \xi, x, \eta) = F(z, \xi, \eta, x)$ and $D(z, \xi, x, \eta) = H(z, \xi, \eta) + C'(z, \xi, \eta)$. Indeed, by Proposition 3.1 (G, D) is a β-pair with $\beta = \alpha_2$. Also, for all $(z, \xi) \in E \times K$ the set $S_\beta(z, \xi)$ is convex (and hence, it is acyclic). Indeed, in our case

$$\begin{aligned} S_\beta(z, \xi) &= \{x \in A(z, \xi) : F(z, \xi, \eta, x) \subset H(z, \xi, \eta) + C'(z, \xi, \eta), \quad \forall \eta \in A(z, \xi)\} \\ &= \bigcap_{\eta \in A(z,\xi)} S(z, \xi, \eta) \end{aligned}$$

where

$$S(z, \xi, \eta) = \{x \in A(z, \xi) : F(z, \xi, \eta, x) \subset H(z, \xi, \eta) + C'(z, \xi, \eta)\}$$

is a convex set (see the second claim of Proposition 2.3). Since $S_\beta(z, \xi)$ is the intersection of a family of convex sets, it must be convex, as required.

It remains to verify that, for fixed $(z, \xi) \in E \times K$, the set $a = A(z, \xi)$ and the maps $f = f_{z,\xi}$, $c = c_{z,\xi}$, $g = g_{z,\xi}$ and $d = d_{z,\xi}$ satisfy all conditions of Propositions 2.1 and 2.2 with $\alpha = \alpha_1$ and $\beta = \alpha_2$. Indeed, condition (i) (resp. (iv)) of Proposition 2.1 is exactly condition (i) (resp. (iv)) of Theorem 4.1 with fixed $(z, \xi) \in E \times K$. Condition (ii) of Proposition 2.1 is derived from the second claim of Corollary 3.2. For the validity of condition (iii) of Proposition 2.1, see the second claim of Proposition 2.3. Condition (i) (resp. (iii)) of Proposition 2.2 is exactly condition (ii) (resp. (iv)) of Theorem 4.1. To verify condition (ii) of Proposition 2.2 it suffices to show that condition (ii)$'$ in Remark 2.3 is fulfilled. In other words, assuming that $(x, \eta) \in a \times a$ with $x \neq \eta$, $\beta(g(x, u), d(x, u))$ and $\overline{\alpha}(f(u, \eta), c(u, \eta))$, i.e., in our case

$$F(z, \xi, u, x) \subset H(z, \xi, u) + C'(z, \xi, u), \tag{4.1}$$

$$F(z, \xi, u, \eta) \subset H(z, \xi, u) + \operatorname{int} C'(z, \xi, u), \tag{4.2}$$

we need to prove that $\overline{\alpha}(f(u, u), c(u, u))$ for all $u \in]x, \eta[$. Indeed, setting $\widetilde{f} = F(z, \xi, u, \cdot)$, $h = H(z, \xi, u)$ and $c' = C'(z, \xi, u)$ we derive from (4.1) and (4.2) that $x \in Q_{\alpha_2}$ and $\eta \in q_{\alpha_2}$ where

$$\begin{aligned} Q_{\alpha_2} &= \{x \in a : \widetilde{f}(x) \subset h + c'\}, \\ q_{\alpha_2} &= \{x \in a : \widetilde{f}(x) \subset h + \operatorname{int} c'\}. \end{aligned}$$

By Proposition 2.3 $u \in q_{\alpha_2}$ for all $u \in]x, \eta[$. In other words,

$$F(z, \xi, u, u) \subset H(z, \xi, u) + \operatorname{int} C'(z, \xi, u),$$

i.e.,

$$\overline{\alpha}(f(u,u), c(u,u)).$$

This proves that condition (ii)$'$ in Remark 2.3 is fulfilled, as desired. □

Theorem 4.2 *Assume that* $F : W \longrightarrow 2^Y$ *is a usc and compact-valued map. Assume additionally that*

(i) For all $(z,\xi) \in E \times K$ *and* $(x,\eta) \in A(z,\xi) \times A(z,\xi)$,

$$F(z,\xi,x,\eta) \subset H(z,\xi,x) + C'(z,\xi,x)$$
$$\Longrightarrow F(z,\xi,\eta,x) \not\subset H(z,\xi,\eta) + \operatorname{int} C'(z,\xi,\eta).$$

(ii) For all $(z,\xi) \in E \times K$ *and* $(x,\eta) \in A(z,\xi) \times A(z,\xi)$ *with* $x \neq \eta$, *if* $F(z,\xi,x,\eta) \not\subset H(z,\xi,x) + C'(z,\xi,x)$ *then* $F(z,\xi,u,\eta) \not\subset H(z,\xi,u) + C'(z,\xi,u)$ *for some* $u \in]x,\eta[$.

(iii) For all $(z,\xi,x) \in E \times K \times K$, $F(z,\xi,x,\cdot)$ *is generalized proper* $[-\operatorname{int} C'(z,\xi,x)]$*-quasiconcave on* $A(z,\xi)$.

(iv) For all $(z,\xi) \in E \times K$ *and* $x \in A(z,\xi)$,

$$F(z,\xi,x,x) \subset H(z,\xi,x) + C'(z,\xi,x).$$

(v) The map

$$(z,\xi,\eta) \in E \times K \times K \longmapsto H(z,\xi,\eta) + \operatorname{int} C'(z,\xi,\eta)$$

has open graph.

Then there exists a solution of Problem (P_{α_2}) *with* C *being defined by* (3.14).

Theorem 4.3 *Assume that* $F : W \longrightarrow 2^Y$ *is a lsc map. Assume additionally that*

(i) For all $(z,\xi) \in E \times K$ *and* $(x,\eta) \in A(z,\xi) \times A(z,\xi)$,

$$F(z,\xi,x,\eta) \cap [H(z,\xi,x) + C'(z,\xi,x)] \neq \emptyset$$
$$\Longrightarrow F(z,\xi,\eta,x) \cap [H(z,\xi,\eta) + \operatorname{int} C'(z,\xi,\eta)] = \emptyset.$$

(ii) For all $(z,\xi) \in E \times K$ *and* $(x,\eta) \in A(z,\xi) \times A(z,\xi)$ *with* $x \neq \eta$,

$$\begin{aligned} &\textit{if} \quad F(z,\xi,x,\eta) \cap [H(z,\xi,x) + C'(z,\xi,x)] = \emptyset \\ &\textit{then} \quad F(z,\xi,u,\eta) \cap [H(z,\xi,u) + C'(z,\xi,u)] = \emptyset, \end{aligned}$$

for some $u \in]x,\eta[$.

(iii) For all $(z,\xi,x) \in E \times K \times K$, $F(z,\xi,x,\cdot)$ *is proper* [*int* $C'(z,\xi,x)$]*-quasiconvex on* $A(z,\xi)$.

(iv) For all $(z,\xi) \in E \times K$ *and* $x \in A(z,\xi)$,

$$F(z,\xi,x,x) \cap [H(z,\xi,x) + C'(z,\xi,x)] \neq \emptyset.$$

(v) The map

$$(z,\xi,\eta) \in E \times K \times K \longmapsto H(z,\xi,\eta) + \operatorname{int} C'(z,\xi,\eta)$$

has open graph.

Then there exists a solution of Problem (P_{α_3}) *with* C *being defined by* (3.14).

Theorem 4.4 *Assume that* $F : W \longrightarrow 2^Y$ *is a usc and compact-valued map. Assume additionally that*

(i) For all $(z, \xi) \in E \times K$ *and* $(x, \eta) \in A(z, \xi) \times A(z, \xi)$,

$$F(z, \xi, x, \eta) \cap [H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)] = \emptyset$$
$$\Longrightarrow F(z, \xi, \eta, x) \cap [H(z, \xi, \eta) + C'(z, \xi, \eta)] \neq \emptyset.$$

(ii) For all $(z, \xi) \in E \times K$ *and* $(x, \eta) \in A(z, \xi) \times A(z, \xi)$ *with* $x \neq \eta$,

$$\begin{aligned} &\textit{if} \quad F(z, \xi, x, \eta) \cap [H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)] \neq \emptyset \\ &\textit{then} \quad F(z, \xi, u, \eta) \cap [H(z, \xi, u) + \operatorname{int} C'(z, \xi, u)] \neq \emptyset, \end{aligned}$$

for some $u \in]x, \eta[$.

(iii) For all $(z, \xi, x) \in E \times K \times K$, $F(z, \xi, x, \cdot)$ *is* $C'(z, \xi, x)$*-concave on* $A(z, \xi)$.

(iv) For all $(z, \xi) \in E \times K$ *and* $x \in A(z, \xi)$,

$$F(z, \xi, x, x) \cap [H(z, \xi, x) + \operatorname{int} C'(z, \xi, x)] = \emptyset.$$

(v) The map

$$(z, \xi, \eta) \in E \times K \times K \longmapsto H(z, \xi, \eta) + C'(z, \xi, \eta)$$

has closed graph.

Then there exists a solution of Problem (P_{α_4}) *with* C *being defined by* (3.13).

Remark 4.1 Existence results for each of Problems (P_{α_i}), $i = 1, 2, 3, 4$, are obtained in [17]. Unfortunately, as we have seen in Sect. 3, they are incorrect. Our Theorems 4.1–4.4 provide exact existence results for these problems under suitable assumptions which are quite different from those of [17]. Observe that Theorems 4.1–4.4 use some pseudomonotonicity type assumptions (see condition (i) in each of these theorems). Existence results in non-monotonicity/non-pseudomonotonicity cases can be found in [24] (see Theorems 3.2–3.5 of [24] and Remarks on pages 620, 622, 623 and 625 of [24]) for problems where E and K are metrizable and conditions (3.13) and (3.14) are simplified: in [24] it is assumed that $H(z, \xi, x) \equiv \{0\}$ and $C'(z, \xi, x)$ does not depend on (z, x). The reader is also referred to [7,22,26,28,44] for the non-monotonicity/non-pseudomonotonicity cases with Y being topological vector spaces [22,26] or the extended real line [7,28,44].

Remark 4.2 The pseudomonotonicity type assumptions in Theorems 4.1–4.4 are inspired by the corresponding notions of pseudomonotonicity of [35] (see [35,Theorem 1, condition (i)] and [35,Theorem 1A, condition (i)]). Another type of pseudomonotonicity assumption which is quite different from [35,Theorem 1, condition (i)] and from our condition (i) in Theorem 4.1 can be found in [34,Theorem 1, condition (iv)]. Pseudomonotonicity assumptions which are different from the corresponding ones of [35], but similar to condition (iv) of Theorem 1 of [34], are also introduced in [1,2,8,10,19,25,30,38] to deal with some simplified versions of Problem (P_{α_1}). So, the results of these papers and the ones of our Theorem 4.1 are different since the pseudomonotonicity type assumptions used in these papers and in our Theorem 4.1 are not the same.

Remark 4.3 Our Theorem 4.1 is different from Corollary 1 of [35]. Indeed, firstly in [35,Corollary 1] it is assumed that $A(z, \xi) \equiv K$ (i.e. A is a constant set-valued map), $H(z, \xi, x) \equiv \{0\}$, and $F(z, \xi, x, \eta) \equiv F(x, \eta)$ and $C'(z, \xi, x) \equiv C'(x)$ (i.e. both the maps F and C' do not depend on (z, ξ)), while in our Theorem 4.1 all these conditions are not required to be satisfied. Secondly, if we restrict ourselves to Problem (P_{α_1}) under the just mentioned conditions of [35,Corollary 1] (and under the compactness of the set K), then we can see that different hypotheses of $F(x, \eta)$ are used in Corollary 1 of [35] and in our Theorem 4.1. Namely, in Theorem 4.1 $F(x, \eta)$ is assumed to be lsc in two variables x and η but the compactness of $F(x, \eta)$ is not required to be satisfied at any point (x, η) of $K \times K$, while in [35,Corollary 1] it is assumed that, for all $(x, \eta) \in K \times K$ with $x \neq \eta$, $F(x, \cdot)$ is lsc, and $F(\cdot, \eta)$ is usc and compact-valued on $[x, \eta]$.

Remark 4.4 Remark 4.3 indicates the difference of our Theorem 4.1 and Corollary 1 of [35]. Similar difference can be shown between our Theorem 4.4 and the corresponding result of [35] dealing with a simplified version of Problem (P_{α_4}). Observe that Problems (P_{α_2}) and (P_{α_3}) are not considered in [35] and hence, the paper [35] does not contain results similar to our Theorems 4.2 and 4.3.

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